$$=\sqrt{\frac{(mt-pt-ns+qs)^2+(sl-m+p)^2+(tl-n+q)^2}{1+s^2+t^2}}.$$

In the problem, $m=5$, $n=6$, $l=7$, $s=2$, $t=-3$, $p=-3$, $q=1$.

$$\therefore p=\sqrt{\frac{(34)^2+(6)^2+(26)^2}{14}}=\tfrac{1}{7}\sqrt{(6538)}=11.55.$$

Also solved by Mary R. Beck, A. H. Holmes, and J. Scheffer, and the Proposer.

307. Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.

A family of planes containing a common line intersects a sphere. Find the orthogonal trajectories of the traces. An analytic solution is preferred.

Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Let the sphere and family of planes be respectively,

$$x^2+y^2+z^2-2az=0,\ z+c=\lambda x\dots(1),$$

in which λ is the parameter of the planes, so that the common line is

$$z+c=x=0\dots(2).$$

The following equations define an inversion in space,

$$x=\frac{2a^2x'}{x'^2+y'^2+z'^2},\quad y=\frac{2a^2y'}{x'^2+y'^2+z'^2},\quad z=\frac{2a^2z'}{x'^2+y'^2+z'^2}\dots(3).$$

The result of (3) applied to (1) is, after reduction,

$$z'-a=0,\quad c(x'^2+y'^2)+ca^2+2a^3-2a^2\lambda x'=0\dots(4).$$

Thus the circles defined by (1) have become the circles in (4), all lying in one plane. The family of circles orthogonal to (4) is

$$z'-a=0,\quad c(x'^2+y'^2)-ca^2-2a^3-2a^2\mu y'=0\dots(5),$$

in which μ is the parameter.

Solving (3) for x', y', z', gives formulas for inverting (4) and (5). The former, of course, becomes (1) again, while the latter becomes

$$x^2+y^2+z^2-2az=0,\quad z(a+c)-ac+\mu ay=0\dots(6).$$

It is a property of inversion that angles remain invariant, so that the system (1) and (6) is orthogonal and (6) is the desired trajectory, composed of circles of which the respective planes have the common line

$$z(a+c)-ac=y=0\dots(7).$$

The inversion of (3) was so chosen that the original sphere became a plane, thus making the solution depend upon the simpler problem of finding the orthogonal trajectory of a family of plane curves.

Also solved by G. B. M. Zerr.

CALCULUS.

81. Proposed by J. OWEN MAHONEY, M. Sc., Dallas, Texas.

Solve, $y^2\dfrac{d^2y}{dx^2}+a\dfrac{dy}{dx}=bx$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $y=A+Bx+Cx^2+Dx^3+Ex^4+\ldots$

$dy/dx=B+2Cx+3Dx^2+4Ex^3+55Fx^4+\ldots$

$d^2y/dx^2=2C+6Dx+12Ex^2+20Fx^3+30Gx^4+\ldots$

$y^2=A^2+B^2x^2+C^2x^4+2ABx+2ACx^2+2ADx^3+2AEx^4+2BCx^3$
$+2BDx^4+\ldots$

$\therefore y^2(d^2y/dx^2)+a(dy/dy)^2=bx$ gives us

$$
\begin{array}{l|l|l|l|l|l}
bx=2A^2C & +6A^2D & x+12A^2E & x^2+20A^2F & x^3+30A^2G & x^4+\ldots \\
\quad +aB^2 & +4ABC & +\ 2B^2C & +\ 6B^2D & +12B^2E & \\
 & +4aBC & +12ABD & +24ABE & +\ 2C^3 & \\
 & & +\ 4AC^2 & +16ACD & +40ABF & \\
 & & +\ 4aC^2 & +\ 4BC^2 & +28ACE & \\
 & & +\ 6aBD & +\ 8aBE & +12AD^2 & \\
 & & & +12aCD & +16BCD & \\
 & & & & +\ 9aD^2 & \\
 & & & & +10aBF & \\
 & & & & +16aCE &
\end{array}
$$

Equating like powers of x we get

$$C=-\frac{aB^2}{2A^2},\quad D=\frac{2A^2b+4aAB^3+4a^2B^3}{12A^4},$$

$$E=\frac{aB^4[A^2-(A+a)\ (4A+3a)]-abA^2B-2bA^3B}{12A^6}.$$

$$\therefore y=A+Bx-\frac{aB^2}{2A^2}x^2+\frac{2A^2b+4AB^3+4a^2B^3}{12A^4}x^3$$

$$+\frac{aB^4[A^2-(A+a)(4A+3a)-abA^2B-2bA^3B}{12A^6}x^4+\ldots$$

where A and B are constants of integration.

This solution does not give a unique result.